The Monin-Obukhov Module for FMS

1 The Similarity Theory

Monin-Obukhov similarity (MOS) theory is the standard method for computing surface fluxes from the lowest level winds, temperatures, and tracer mixing ratios in GCMs. The lowest level is assumed to lie within the "surface layer" in which turbulent fluxes have negligible vertical variation, and in which MOS assumes that the wind and buoyancy profiles are a function only of the surface stress, the surface buoyancy flux, and the height z. A good reference is

Garratt, J. R. "The Atmospheric Boundary Layer", Cambridge University Press, 1992

1.1 Scales

The surface stress provides a velocity scale, u_*, defined so that the magnitude of the surface stress τ is given by

$$\tau = \rho_s u_*^2 \tag{1}$$

where ρ_s is the surface air density. The upward surface buoyancy flux B can then be used to define a buoyancy scale, b_*,

$$B = \rho_s u_* b_* \tag{2}$$

In the atmosphere, if one ignores the virtual temperature effect, the buoyancy can be taken to be

$$b = g\frac{(\Theta - \Theta_0)}{\Theta_0} \tag{3}$$

where Θ is the potential temperature and Θ_0 is a constant reference value, which can be chosen equal to the surface value. For a hydrostatic atmosphere,

$$\frac{1}{\Theta}\frac{\partial \Theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} + \frac{g}{c_p} \tag{4}$$

so, instead of potential temperature, one can equally well use the dry static energy divided by c_p: $T + gz/c_p$. To include virtual temperature effects, one replaces T or Θ by the virtual temperature or virtual potential temperature. The effects of the humidity difference between the surface and the lowest model level can be as large as the temperature difference over the tropical oceans, when computing buoyancy gradients.

This scaling for the buoyancy is inappropriate in the "free-convective limit" in which stress, or u_*, vanishes but there is still a non-zero buoyancy flux. We return to the question of the free-convective limit of MOS theory below.

The units of buoyancy are $[meters]/[sec]^2$, so one can create a length scale from u_* and b_*,

$$L = -u_*^2/(\kappa b_*) \tag{5}$$

The inclusion of κ = von Karman's constant is conventional. L is referred to as the Monin-Obukhov length. With this sign convention, L is positive under stable conditions, in which $B < 0$ (heat flux into the surface). Roughly speaking, for $z < |L|$ the turbulence is primarily driven mechanically and for $z > |L|$ it is primarily driven by buoyancy. Mechanically driven turbulence always wins out close enough to the surface, as long as the stress is non-zero.

1.2 The neutral case

In the neutral case, $b_* = 0$, and the wind shear is assumed be a function only of the stress, u_* and the height above the surface z.

$$\frac{\partial u}{\partial z} = \frac{u_*}{\kappa z} \tag{6}$$

This equation can be thought of as defining κ, which is assumed to be a universal constant. From laboratory experiments and atmospheric observations, the value of κ is about 0.4. Integrating, we obtain the famous logarithmic "law of the wall":

$$u(z) = \frac{u_*}{\kappa} ln(\frac{z}{z_0}) \quad (7)$$

where u is the wind component in the direction of the surface stress, and z_0 is the "roughness height" , defined as the height at which the logarithmic profile would yield zero wind. (The turning of the wind is assumed to be negligible within this surface layer.) The roughness height is the key parameter describing the macroscopic effects of the surface type on the stress.

The "neutral drag coefficient" , given the flow at height z, $C_n(z)$, is defined so that, in the neutral case, the surface stress vector is given by

$$\tau = \rho_s C_n |\mathbf{v}|\mathbf{v} \quad (8)$$

$$\frac{|\tau|}{\rho_s} = u_*^2 = C_n(z)|\mathbf{v}(z)|^2 \quad (9)$$

so that

$$C_n(z) = [\frac{\kappa}{ln(z/z_0)}]^2 \quad (10)$$

The neutral drag coefficient is a function of the height at which the winds are supplied. Given the roughness height and the height z of the specified wind, we have simple expressions for the drag coefficient and surface stress. (Quite often in meteorology, the term "surface winds" refers to the winds at a height of 10 m, and drag coefficients are quoted with the assumption that they refer to winds at this height.)

1.3 Stratification

In the stratified case, the key assumption in MOS is that the profile can also depend on $\zeta \equiv z/L$. It is conventional to write

$$\frac{\kappa z}{u_*}\frac{\partial u}{\partial z} = \Phi_m(\zeta) \quad (11)$$

where Φ_m is presumed to be a universal function of ζ. $\Phi_m \to 1$ as $\zeta \to 0$, so that the logarithmic profile is achieved as the surface is approached. Integrating from z_0 to z, we find

$$u(z) = \frac{u_*}{\kappa}(F_m(z/L) - F_m(z_0/L)) \quad (12)$$

where

$$F_m(\zeta) \equiv \int \zeta^{-1}\Phi_m d\zeta \tag{13}$$

The assumption is that z_0 is not a function of stability, which is plausible as long as $z_0 << |L|$.

The problem now becomes that of generating the simultaneous surface fluxes of buoyancy and momentum from the values of u and b at some height z; the determination of the wind profile is coupled to that of determining the buoyancy profile.

Therefore, by the same scaling argument, one writes the buoyancy profile as,

$$\frac{\kappa z}{b_*}\frac{\partial b}{\partial z} = \Phi_b(\zeta) \tag{14}$$

Integrating as before,

$$b(z) - b(0) = \frac{b_*}{\kappa}(F_b(z/L) - F_b(z_b/L)) \tag{15}$$

where z_b is the roughness length for bouyancy, defined so that b would take on its surface value at z_b. In the neutral limit, both F_m and F_b should equal $ln(\zeta)$.

In general, there is no reason to expect z_0 and z_b to be the same. Typically $z_0 > z_b$ because momentum can be transferred to the surface through pressure forces as well as molecular viscosity, whereas temperature and humidity must be transferred by molecular diffusion at the surface (except when there is sea spray etc) One sometimes uses the notation

$$n \equiv ln(z_0/z_b) \tag{16}$$

For example, Garratt suggests that $n \approx 2$ over typical land surfaces.

2 Solving For Surface Fluxes

Given the form of the stability functions F_m and F_b, one can obtain a consistency condition by computing L

$$L = -u_*^2/(\kappa b_*) = \frac{u(z)^2(F_b(\zeta) - F_b(\zeta z_b/z))}{(b(z) - b(0))(F_m(\zeta) - F_m(\zeta z_0/z))^2} \tag{17}$$

or

$$R_b = \zeta \frac{F_b(\zeta) - F_b(\zeta z_b/z)}{(F_m(\zeta) - F_m(\zeta z_0/z))^2} \tag{18}$$

where R_b is the "Bulk Richardson number" between the surface and the height z

$$R_b \equiv \frac{z(b(z) - b(0))}{u(z)^2} = \frac{gz(\Theta_v(z) - \Theta_v(0))}{\Theta_v(0)u(z)^2} \tag{19}$$

Eq (18) can then be solved iteratively for ζ, given R_b, z_0/z, and z_b/z.

Given ζ, one can then compute

$$u_* = \frac{\kappa|u(z)|}{F_m(\zeta) - F_m(\zeta z_0/z)} \tag{20}$$

and

$$b_* = \frac{\kappa(b(z) - b(0))}{F_b(\zeta) - F_h(\zeta z_b/z)} \tag{21}$$

These values correspond to the drag coefficients

$$C_m = \frac{\kappa^2}{(F_m(\zeta) - F_m(\zeta z_0/z))^2} \tag{22}$$

and

$$C_b = \frac{\kappa^2}{(F_m(\zeta) - F_m(\zeta z_0/z))(F_b(\zeta) - F_b(\zeta z_b/z))} \tag{23}$$

In implementing this theory, once one has chosen similarity functions, one has a choice of iterating to a solution whenever it is needed, or of tabulating the solution for the relevant range of inputs, and, potentially, fitting the results with easly evaluated functions. Given that there are three inputs, the typical (but not universal) choice is to iterate. If there were only two inputs - i.e., if one could assume that the ratio of z_0 to z_h were fixed, tabulation and curve fitting would be more straighforward. In FMS, the solution is obtained by iteration.

For the purpose of iteration using Newton-Raphson, we note that the needed derivative is

$$\partial R_b/\partial\zeta = R_b\left(\frac{1}{\zeta} + \frac{\Phi_b(\zeta) - \Phi_b(\zeta z_0/z)}{F_b(\zeta) - F_b(\zeta z_b/z)} - 2\frac{\Phi_m(\zeta) - \Phi_m(\zeta z_0/z)}{F_m(\zeta) - F_m(\zeta z_b/z)}\right) \tag{24}$$

3 The Stability Functions

Ideally there would be a generally accepted theory for the stability functions, but no such theory exists, and they are evaluated from atmospheric field studies. The forms that receive the most support, and that are recommended by Garratt, are for the unstable case $\zeta < 0$,

$$\Phi_m = (1 - 16\zeta)^{-1/4} \tag{25}$$

$$\Phi_b = (1 - 16\zeta)^{-1/2} \tag{26}$$

and in the stable case $\zeta > 0$,

$$\Phi_m = \Phi_b = 1 + 5\zeta \tag{27}$$

These are considered to have empirical support in the range $-5 < \zeta < 1$.

3.1 The unstable case

Consider the limit that the mean wind, and therefore, the surface stress and the friction velocity u_* tend to zero. With the choice of stability functions presented above, the result will be zero buoyancy flux as well. This will be true as long as $\Phi_b \to 0$ more rapidly than $\zeta^{-1/3}$ as $|\zeta| \to \infty$.

To see why $\Phi_b \propto |\zeta|^{-1/3}$ is special, look at the buoyancy profile and ask under what conditions it approaches a well-defined limit as $u_* \to 0$. We can rewrite (14) as

$$\frac{\partial b}{\partial z} = \frac{B}{\kappa u_* z}\Phi_b(\zeta) = \frac{B}{\kappa u_* z}\Phi_b\left(\frac{-\kappa z b_*}{u_*^2}\right) = \frac{B}{\kappa u_* z}\Phi_b\left(\frac{-\kappa z B}{u_*^3}\right) \tag{28}$$

where B is here the buoyancy flux divided by the surface density. In order for this profile to be well-defined as $u_* \to 0$ and $\zeta \to -\infty$, we need $\Phi_b \propto |\zeta|^{-1/3}$ so that the factors of u_* cancel, leading to

$$\frac{\partial b}{\partial z} \propto \frac{B^{2/3}}{z^{4/3}} \tag{29}$$

which is referred to as the free-convection limit.

With the recommended stability function (26) we have instead $\Phi_b \propto |\zeta|^{-1/2}$ for large $|\zeta|$. To see that this results in zero buoyancy flux in the free convective limit, we note that in the large $|\zeta|$ limit we have

$$B = \frac{\kappa^2 \Delta b |\mathbf{v}|}{F_m(\zeta) F_b(\zeta)} \tag{30}$$

where Δb is the buoyancy difference between the surface and level z and ζ is determined by

$$R_b = \frac{z\Delta b}{|\mathbf{v}|^2} = \frac{\zeta F_b(\zeta)}{F_m(\zeta)^2} \tag{31}$$

Eliminating F_m we have

$$B^2 \propto \frac{z(\Delta b)^3}{\zeta F_b(\zeta)^3} \tag{32}$$

If $\Phi_b \propto |\zeta|^{-1/2}$ for large $|\zeta|$ then we also have $F_b \propto |\zeta|^{-1/2}$. So $B \to 0$ as $|\zeta| \to \infty$.

This is unsatisfactory – either one must use a stability function that is consistent with the free-convective limit, or one must not allow the wind speed that is used in the similarity theory to approach 0. The latter alternative is the choice made in our model and in many others. The picture is that in the convective limit, eddies transporting buoyancy across level z have length scales that scale not with z but with the depth of the convective layer h, which violates the assumptions of the original similarity theory. These eddies produce wind speed perturbations ("gusts") of magnitude

$$G = (Bh)^{1/3} \tag{33}$$

which is the only velocity scale that one can generate from B and h. One can now visualize using the similarity theory on smaller scales than these gusts, implying that the gusts must be included in the wind speed that is input into MOS. From the viewpoint of the MOS module itself, it need not be concerned with the free-convective limit – it assumes that the input wind speed will be bounded away from zero by some theory of "gustiness".

The stability functions (25) and (26) have the property that $\Phi_m^2 = \Phi_h$. This implies that the local (as opposed to the bulk) Richardson's number is equal to ζ, and, in particular, that Ri is a linear function of z. If there were some dynamical argument for this simple structure, it would help in justifying this choice of stability functions, but we are not aware of such an argument.

These forms have the convenient property that we can evaluate the integral stability functions, F_m and F_b analytically. One finds, with considerable effort, that (using the notation $F(\zeta;\zeta_0) = F(\zeta) - F(\zeta_0)$ where $\zeta_0 = z_0/L = \zeta z_0/z$)

$$F_m(\zeta;\zeta_0) = \ln(\frac{z}{z_0}) - 2\ln(\frac{1+x}{1+x_0}) - \ln(\frac{1+x^2}{1+x_0^2}) + 2(\tan^{-1}(x) - \tan^{-1}(x_0)) \quad (34)$$

where $x \equiv (1-16\zeta)^{1/4}$; while for buoyancy

$$F_b(\zeta;\zeta_b) = \ln(\frac{z}{z_b}) - 2\ln(\frac{1+y}{1+y_b}) \quad (35)$$

where $y \equiv (1-16\zeta)^{1/2}$.

These expressions are sufficiently costly to compute, especially F_m, that there might be value in using a table look up for them, even as part of the overall iteration. At present, they are simply computed directly from these expressions within FMS.

3.2 The stable case

It is not difficult to show that with $\Phi_b \equiv \Phi_m$, and, therefore, $F_b \equiv F_m$ on the stable side that the drag coefficients drop to zero close to $Ri_b < 0.2$, given the form (27). This critical Richarsons's number would be exactly 0.2 if one set $z_0 = z_h$. To see this, note that, with (27) we have $F_b = F_m = ln(\zeta) + \beta\zeta$ with $\beta = 5$, so that

$$R_b = \frac{\zeta}{ln(z/z_0) + \beta(\zeta - \zeta_0)} \quad (36)$$

As ζ increases, this in bounded above by $R_{crit} = \beta^{-1}$. Therefore, as R_b approaches R_{crit}, $\zeta \to \infty$, and, therefore, $F_m, F_t \to \infty$ as well., implying that the drag coefficients $\to 0$

While 0.2 is close to the 0.25 stability criterion for linear Kelvin Helmholtz instability, it is inconsistent with field data to set the fluxes to zero at this low a bulk Richardson's number. Intermittent turbulence still exists at higher stability, associated with breaking gravity waves likely related to inhomogeneities in the surface whose effects are not representable in terms of a surface roughness.

Since the value $\beta = 5$ seems to agree with data in the moderately stable range, one generally modifies Φ_b and Φ_m so that they have this form for small ζ but allow non-zero fluxes for larger Ri_b. Different model use various forms for this purpose. The simplest would seem to be the most desirable. It is also very convenient to have a form that allows one to integrate $\Phi_m(\zeta)/\zeta$ analytically so that one has a relatively simple expression for F_m. The MOS module has two options:

Version 1 is

$$\Phi_m = \Phi_b = 1 + \zeta \frac{5 + \beta\zeta}{1 + \zeta} \tag{37}$$

This approaches $1 + \beta\zeta$ for large ζ while preserving the observed small ζ behavior. One can use β as a tunable parameter to modify the poorly understood surface mixing in very stable conditions. In the code, the namelist parameter is rich_crit $\equiv 1/\beta$

Version 2, which we now favor, in that it provides additional controls on the mixing, is piecewise linear

$$\Phi_m = \Phi_b = 1 + 5\zeta; \quad \zeta < \zeta_T \tag{38}$$

$$\Phi_m = \Phi_b = 1 + (5 - \beta)\zeta_c + \beta\zeta; \quad \zeta \geq \zeta_T \tag{39}$$

Here β controls the critical Ri as before, while ζ_T controls the point at which a transition is made from the established stability function for the fully tubulent boundary layer to the presumably intermitent turbulence at high stability. With this version one can generate drag coefficients that are small but non-zero for a large range of Ri (see the following section). In the namelist, ζ_T is denoted by zeta_trans.

The integrals of $\Phi(\zeta)/\zeta$ can once again be computed analytically.

4 Diffusion coefficients

The profiles of wind and buoyancy obtained in this way can be thought of as determining diffusivities that would result in these profiles if vertical diffusion were the only process acting – which the surface flux similarity theory effectively assumes in any case. To obtain this equivalent diffusivity, one can equate the surface flux with the diffusivity multiplied by the gradient. In particular, the kinematic diffusivity for momentum K_m is then

$$K_m = \frac{u_*^2}{\partial_z u} = \frac{\kappa u_* z}{\Phi_m(z/L)} \tag{40}$$

The diffusivity for buoyancy (ie potential temperature or dry static energy) is

$$K_b = \frac{u_* b_*}{\partial_z b} = \frac{\kappa u_* z}{\Phi_b(z/L)} \tag{41}$$

corresponding to a Prandtl ratio, $Pr \equiv K_m/K_b = \Phi_b/\Phi_m$.

If one's GCM has sufficient vertical resolution to penetrate this surface layer, and assuming that the GCM is diffusing in the vertical, then its diffusivity and viscosity should approach these values as the surface is approached.

In the stable planetary boundary layer above the surface layer, the diffusivity in GCMs is often modeled as being local in the vertical, of the form

$$K = \ell^2 |\frac{\partial u}{\partial z}| \tag{42}$$

where ℓ is a mixing length. In the neutral surface layer, we have

$$K_m = K_b = \frac{u_*^2}{|\partial u/\partial z|} = (\kappa z)^2 |\frac{\partial u}{\partial z}| \tag{43}$$

so in this limit, $\ell = \kappa z$. In the stratified case, we write

$$\ell = \kappa z f(Ri)^{1/2} \tag{44}$$

so that

$$K = (\kappa z)^2 |\frac{\partial u}{\partial z}| f(Ri) \tag{45}$$

In terms of our stability function Φ, assumed to be the same for momentum and buoyancy, we have for the stratified case

$$K_m = K_b = \frac{(\kappa z)^2}{\Phi^2} |\frac{\partial u}{\partial z}| \tag{46}$$

Therefore, we can set

$$f(Ri) = (\Phi(\zeta))^{-2} \tag{47}$$

where, we recall, the relationship between Ri and ζ is

$$Ri = \zeta \frac{\Phi_b}{\Phi_m^2} = \frac{\zeta}{\Phi} \tag{48}$$

given the assumed equality of Φ_m and Φ_b. Solving this last expression for ζ as a function of Ri, one can substitute into (47) to obtain f as a function of Ri.

If one simply used $\Phi = 1 + \beta\zeta$ one would find that

$$f(Ri) = (1 - \beta Ri)^2 \tag{49}$$

The analogous expression for version 1 involves the positive solution of the quadratic

$$(Ri^{-1} - \beta)\zeta + (Ri^{-1} + 6)\zeta - 1 = 0 \tag{50}$$

One can then set $K = \Phi^{-2}$, where Φ is given by (37).

For version 2 one simply gets

$$f(Ri) = (1 - 5Ri)^2; \quad Ri < Ri_T \tag{51}$$

and

$$f(Ri) = \left(\frac{1 - \beta Ri}{1 + (5 - \beta)\zeta_T} \right)^2; \quad Ri > Ri_T \tag{52}$$

where

$$Ri_T \equiv \frac{\zeta_T}{1 + 5\zeta_T} \tag{53}$$

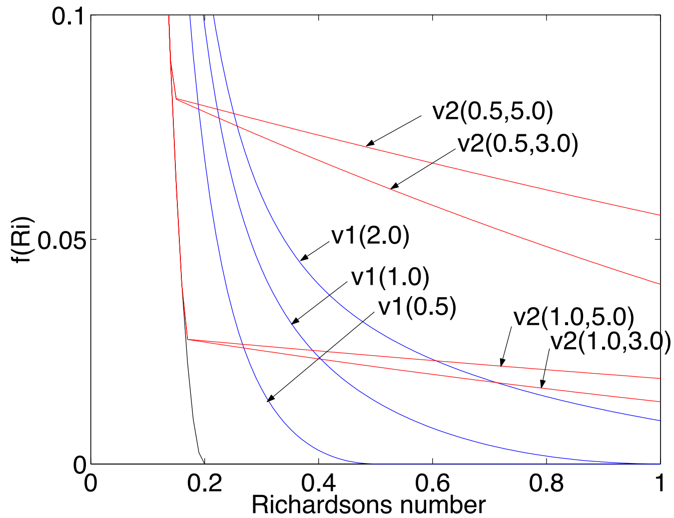

Figure 1: $f(Ri)$ as a function of Ri. The notation is $v1(x) \rightarrow$ version 1 with $x = Ri_{crit} = 1/\beta$, $v2(x, y) \rightarrow$ version 2 with $x = \zeta_T$ and $y = Ri_{crit}$

One can see from the figure that version 2 allows one to match the empirically observed stability function for weakly stable conditions while still being able to tune the mixing at large stability, for which there is little empirical guidance and for which MOS is presumably invalid in any case. In version 1, if one tries to increase the mixing at high stabilities one begins to depart substantially from the empirical relation at weak stability.

In addition to providing a way of matching an interior diffusivity of the form (45), these expressions also provide a useful way of thinking about the strength of the mixing in the surface layer implied by the similarity profiles.

5 Implementation Notes

Four interfaces are provided (see FMS documentation for details)

- **mo_drag** returns drag coefficients for momentum, heat, and specific humidity using as input the height z, the effective wind speed at z

(incorporating gustiness), and the virtual potential temperature at z and at the surface. In addition to the three drag coefficients, output also includes u_* and b_*

- **mo_profile** returns the profiles of momentum, heat, and specific humidity below height z, given the three roughness lengths as well as $u*$, $b*$, and $q*$. (while u_* and b_* are available as output from mo_drag, q_* must be computed by dividing the evaporation by u_*)
- **mo_diff** returns the diffusivities consistent with the similarity profiles, as described by Eqs. (40) and (41), given u_* and b_*
- **stable_mix** returns the function $f(Ri)$ described by Eq. (45)

The relevance of this theory to boundary layer fluxes is typically limited to $\approx 10\%$ of the planetary boundary layer depth (PBL). This number seem to arise from the fact that O(1) wind turning and stress changes occur on the scale of the PBL, so these are likely to be O(1/10) over ther lowest 10 percent of the PBL.) This provides strong motivation to place the lowest model level close to the surface, preferably less than 50 meters.

The assumption is that the differential stability functions for tracers, such as specific humidity, are the same as those for buoyancy, or potential temperature. However, the roughness length for tracers can be different from that for buoyancy or momentum, so that the final profiles, and drag coeffcients, can be different. The monin-obukhov length L and the drag coeffcients for momentum and buoyancy (heat) are functions of the momentum and buoyancy roughness lengths. The The drag coefficient for tracers is then a function of L and the roughness length for tracer.

The convergence criterion is that the change in ζ from the previous iteration $\delta\zeta$, satisfies $min(|\delta\zeta|, |\delta\zeta/\zeta|) < 1.e - 04$

Convergence of the interation can be slow when R_b is close to R_{crit}. To avoid this, Ri_b is arbitrarily set to Ri_{crit} whenever $Ri > 0.95 R_{crit}$

The value of von_Karman's constant is taken from constants_mod, as is the value of g, required to compute buoyancy from the potential temperatures input into mo_drag.