Let M be a Riemannian manifold embedded in $\mathbb{R}^{\hat{m}}$ by $h_M \colon M \to \mathbb{R}^{\hat{m}}$ and N be a Riemannian manifold embedded in $\mathbb{R}^{\hat{n}}$ by $h_N \colon N \to \mathbb{R}^{\hat{n}}$. Let p be a point on M with a neigborhood $M_p \subseteq M$. Let $f \colon M_p \to N$ be a C^1 map. The embeddings are not necessarily isometric but they must be at least C^1 and invertible in a neighborhood of p (or, respectively, $f(p)$.

The differential $\mathrm{d}f_p \colon T_pM \to T_{f(p)}N$ is a function such that for any curve $\gamma \colon \mathbb{R} \to M$, $\gamma(0) = p$, $\gamma'(0) = v$, γ corresponds to the tangent vector $v \in T_pM$, we have

$$\mathrm{d}f_p(\gamma'(0)) = (f \circ \gamma)'(0). \tag{1}$$

For example $\gamma(t) = \exp_p(tv)$. Thus

$$\mathrm{d}f_p(v) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp_p(tv))(0). \tag{2}$$

This can also be expressed as

$$\mathrm{d}f_p(v) = \log_{f(p)}(f(\exp_p(v))) \tag{3}$$

from the definition of the logarithmic map. This is true as long as all functions are defined, and they are for sufficiently small vectors v.

We can represent the function f as a mapping between subsets of the spaces M_p and N are embedded in, $\hat{f} \colon h_M^{-1}(M_p) \to \mathbb{R}^{\hat{n}}$:

$$\hat{f}(x) = h_N(f(h_M^{-1}(x))) \tag{4}$$

for any $x \in h_M(M_p)$.

Similarly, let us assume that T_pM is embedded by h_{T_pM} as a linear subspace U_M of $\mathbb{R}^{\hat{m}}$ and $T_{f(p)}N$ is embedded by $h_{T_{f(p)}N}$ as a linear subspace U_N of $\mathbb{R}^{\hat{n}}$. Using these, we can represent the differential $\mathrm{d}f_p$ in the embedding by $\mathrm{d}\hat{f}_p \colon U_M \to U_N$:

$$\mathrm{d}\hat{f}_p(v) = h_{T_{f(p)}N}(\mathrm{d}f_p(h_{T_pM}^{-1}(v))) \tag{5}$$

for any $v \in U_M$. In this setting $\mathrm{d}\hat{f}_p$ is just a linear transformation between two vector subspaces. It is thus completely determined by, for example, its values on a basis of U_M.

Substituting everything we get

$$\mathrm{d}\hat{f}_p(v) = (h_{T_{f(p)}N} \circ \log_{f(p)} \circ h_N^{-1} \circ \hat{f} \circ h_M \circ \exp_p \circ h_{T_pM}^{-1})(v) \tag{6}$$

which might look useless until we notice that we can calculate values of $h_{T_{f(p)}N} \circ \log_{f(p)} \circ h_N^{-1}$, $\hat{f}$ and $h_M \circ \exp_p \circ h_{T_pM}^{-1}$ easily in our computer programs.

One way forward now is to put different vectors v to Eq.(6) and see what is returned. We could, however, take a basis $v_1, v_2, \ldots, v_m$ of U_M, define

$$g(t_1, t_2, \ldots, t_m) = \mathrm{d}\hat{f}_p \left(\sum_{i=1}^{m} t_i v_i \right) \tag{7}$$

and calculate Jacobian of g at zeros using automatic differentiation to get an easy method of computing $\mathrm{d}\hat{f}_p(v)$.

Alternatively, we could take a basis $v_1, v_2, \ldots, v_{\hat{m}}$ of $\mathbb{R}^{\hat{m}}$, define

$$g_2(t_1, t_2, \ldots, t_{\hat{m}}) = \mathrm{d}\hat{f}_p\left(\sum_{i=1}^{\hat{m}} t_i v_i\right) \tag{8}$$

and calculate Jacobian of g_2. As long as the it is given a vector from U_M the expected result will be returned, although care must be taken to avoid giving g_2 coefficients t_i that do not correspond to a vector from U_M.